

Bayesian Econometrics

Conjugate Families

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Outline

1 Conjugate Families

Conjugate Families

- A Conjugate family is a kind of prior distributions for which the posterior distribution can be easily calculated.
- Conjugate prior distributions play an important role in Bayesian methods, because their use can simplify the process of integration required for marginalization.
- When a prior and posterior are from a same family, the actualization process for parameters is simplified.

Conjugate Families

Binomial distribution

Theorem

Suppose that $X_1 \dots X_n$ is a random sample from a Bernoulli distribution with parameter p , which is unknown. Also assume that the prior distribution for p is beta with parameters $\alpha(> 0)$ and $\beta(> 0)$. Then the posterior distribution for p , is beta with parameters $\alpha + \sum X_i$ and $\beta + n - \sum X_i$.

Conjugate Families

Binomial distribution

$$\begin{aligned}\pi(p) &\propto p^{\alpha-1}(1-p)^{\beta-1}, \\ L(p|X) &\propto p^{\sum X_i}(1-p)^{n-\sum X_i}\end{aligned}$$

then,

$$\begin{aligned}\pi(p|X) &\propto \pi(p)L(p|X) \\ \pi(p|X) &\propto p^{\alpha+\sum X_i-1}(1-p)^{\beta+n-\sum X_i-1}.\end{aligned}$$

We can see that: $p|x \sim \text{beta}(\alpha + \sum X_i; \beta + n - \sum X_i)$.

Conjugate Families

Poisson distribution

Theorem

Suppose that $X_1 \dots X_n$ is a random sample from a Poisson distribution with unknown mean λ . Also assume that the prior distribution for λ is Gamma with parameters $\alpha(> 0)$ and $\beta(> 0)$. Then the posterior distribution for λ is Gamma with parameters $\alpha + \sum X_i$ and $\beta + n$.

Conjugate Families

Poisson distribution

$$\begin{aligned}\pi(\lambda) &\propto \lambda^{\alpha-1} e^{-\lambda\beta}, \\ L(\lambda|X) &\propto e^{-n\lambda} \lambda^{\sum X_i},\end{aligned}$$

then,

$$\begin{aligned}\pi(\lambda|X) &\propto \pi(\lambda)L(\lambda|X) \\ \pi(p|X) &\propto \lambda^{\alpha+\sum X_i-1} e^{-\lambda(\beta+n)}.\end{aligned}$$

We can see that: $\lambda|X \sim \text{Gamma}(\alpha + \sum X_i; \beta + n)$.

Conjugate Families

Exponential distribution

Theorem

Suppose that $X_1 \dots X_n$ is a random sample from an Exponential distribution with unknown mean λ . Also assume that the prior distribution for λ is Gamma with parameters $\alpha(> 0)$ and $\beta(> 0)$. Then the posterior distribution for λ is Gamma with parameters $\alpha + n$ and $\beta + \sum X_i$.

Conjugate Families

Exponential distribution

$$\begin{aligned}\pi(\lambda) &\propto \lambda^{\alpha-1} e^{-\lambda\beta}, \\ L(\lambda|X) &\propto \lambda^n e^{-\lambda\sum X_i},\end{aligned}$$

then,

$$\begin{aligned}\pi(\lambda|X) &\propto \pi(\lambda)L(\lambda|X) \\ \pi(p|X) &\propto \lambda^{\alpha+n-1} e^{-\lambda(\beta+\sum X_i)}.\end{aligned}$$

We can see that: $\lambda|X \sim \text{Gamma}(\alpha + n; \beta + \sum X_i)$.

Conjugate Families

Normal-Gamma

Theorem

Suppose that $X_1 \dots X_n$ is a random sample from a Normal distribution with unknown parameters μ and precision τ . Also assume that the prior distributions for μ and τ are respectively Normal $N(\mu_0, \tau_0)$ and *Gamma* (α_0, β_0) . The posterior distributions for both are respectively $N(\mu_1, \tau_1)$ and *Gamma* (α_1, β_1) , where:

$$\mu_1 = \frac{\tau_0 \mu_0 + n \bar{X}}{\tau_0 + n}; \tau_1 = \tau_0 + n; \alpha_1 = \alpha_0 + n/2 \text{ and}$$

$$\beta_1 = \beta_0 + \frac{1}{2} \sum (X_i - \bar{X})^2 + \frac{\tau_0 n (\bar{X} - \mu_0)^2}{2(\tau_0 - n)}.$$